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# A Quantitative Version of the Dirichlet–Jordan Test for Double Fourier Series

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For classes of functions with convergent Fourier series, the problem of estimating the rate of convergence has always been of interest. The classical theorem of Dirichlet and Jordan for functions of bounded variation assures the convergence of their Fourier series, but gives no estimate of the rate of convergence. Such an estimate was first provided by Bojanić. Here we consider this problem in the case of functions of two variables that are of bounded variation in the sense of Hardy and Krause. The Dirichlet–Jordan test was first extended by Hardy from single to double Fourier series. Now, we provide a quantitative version of it. We prove our estimate in a greater generality, by introducing the so-called rectangular oscillation of a function of two variables over a rectangle. 

## 1. INTRODUCTION: FUNCTIONS OF BOUNDED VARIATION ON THE PLANE

Throughout the section, let $J := [a, b]$ and $K := [c, d]$ be two fixed intervals in $\mathbf{R}$. We remind the reader of an appropriate notion of variation for a complex-valued function defined on $J \times K$. Of the many possible notions (see [1]), the one tailored to the present purpose is that due to Hardy [7] and Krause. (See the discussion in [8, Sect. 254].)

Given two partitions

$$\mathscr{D}_1 : a = x_0 < x_1 < \cdots < x_m = b \qquad \text{and} \qquad \mathscr{D}_2 : c = y_0 < y_1 < \cdots < y_n = b$$

and a function $f: J \times K \to \mathbf{C}$, we form a rectangular grid $\mathscr{D} := \mathscr{D}_1 \times \mathscr{D}_2$ on $J \times K$ and set

$$\mathscr{D}(f) := \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} |f(x_j, y_k) - f(x_{j+1}, y_k) - f(x_j, y_{k+1}) + f(x_{j+1}, y_{k+1})|.$$

We define the (total) variation of $f$ on $J \times K$ by

$$\operatorname{var}_2(f, J \times K) := \sup\{\mathscr{D}(f) : \mathscr{D} \text{ is a rectangular grid on } J \times K\} \tag{1.1}$$



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and say that $f$ is of bounded variation (in the sense of Hardy and Krause) if each of the numbers

$$\mathrm{var}_2(f, J\times K), \qquad \mathrm{var}_1(f(\cdot, c), J), \qquad \mathrm{var}_1(f(a, \cdot), K)$$

is finite. Here the last two quantities are the ordinary variations of the single variable functions $f(x, c)$ and $f(a, y)$, respectively. For instance,

$$\mathrm{var}_1(f(\cdot, c), J) := \sup\{\mathscr{D}_1(f(\cdot, c)) : \mathscr{D}_1 \text{ is a partition of } J\}, \tag{1.2}$$

where

$$\mathscr{D}_1(f(\cdot, c)) := \sum_{j=0}^{m-1} |f(x_j, c) - f(x_{j+1}, c)|;$$

and $\mathrm{var}_1(f(a, \cdot), K)$ is defined analogously.

We denote by $BV(J\times K)$ the collection of all functions $f\colon J\times K \to \mathbf{C}$ of bounded variation. As is known (see [2]), $BV(J\times K)$ is a Banach space with the norm given by

$$|||f||| := |f(a, c)| + \mathrm{var}_1(f(\cdot, c), J) + \mathrm{var}_1(f(a, \cdot), K) + \mathrm{var}_2(f, J\times K).$$

A few remarks about the above definition are in order. Let $f\in BV(J\times K)$. Then it is easily checked that $f$ is bounded on $J\times K$ satisfying

$$\|f\|_\infty := \sup\{|f(x, y)| : (x, y)\in J\times K\} \leqslant |||f|||.$$

Furthermore, for each fixed $x\in J$ and $y\in K$, the marginal functions $f(\cdot, y)$ and $f(x, \cdot)$ are of bounded variation on $J$ and $K$, respectively, with

$$\mathrm{var}_1(f(\cdot, y), J) \leqslant |||f||| \qquad \text{and} \qquad \mathrm{var}_1(f(x, \cdot), K) \leqslant |||f|||.$$

Thus, we can replace $|||\cdot|||$ by many equivalent Banach space norms. For example, the term $f(a, c)$ can be replaced by $f(x, y)$ for any $(x, y)\in J\times K$ or by $\|f\|_\infty$, and the term $\mathrm{var}_1(f(\cdot, c), J)$ by $\mathrm{var}_1(f(\cdot, y), J)$ or $\sup\{\mathrm{var}_1(f(\cdot, y), J) : y\in K\}$, etc.

It is also easily verified that if $f$ is twice continuously differentiable in both variables, then $f\in BV(J\times K)$ and

$$|||f||| = |f(a, c)| + \int_a^b \left|\frac{\partial f(x, c)}{\partial x}\right| dx + \int_c^d \left|\frac{\partial f(a, y)}{\partial y}\right| dy$$
$$+ \int_a^b \int_c^d \left|\frac{\partial^2 f(x, y)}{\partial x\, \partial y}\right| dx\, dy.$$

Finally, analogously to the one-dimensional case, it is also true that the limit

$$f(x+0, y+0) := \lim\{f(x+u, y+v) : u, v\to 0 \text{ and } u, v > 0\}$$

exists for all $(x, y) \in [a, b) \times [c, d)$. Similar statements are true for the limits $f(x-0, y+0)$, $f(x+0, y-0)$, and $f(x-0, y-0)$, as well. Accordingly, if a function $f\colon \mathbf{R}^2 \to \mathbf{C}$ has period $2\pi$ in each variable and is of bounded variation on the two-dimensional torus $\mathbf{T}_2 := [-\pi, \pi] \times [-\pi, \pi]$, then each of the four limits $f(x \pm 0, y \pm 0)$ exists for all $(x, y)$. Concerning these properties, we refer the reader to consult with [5–8].

## 2. Double Fourier Series and Preliminary Results

Let $f\colon \mathbf{R}^2 \to \mathbf{C}$ be a function, $2\pi$-periodic in each variable and integrable over $\mathbf{T}^2$. We remind the reader that the double Fourier series of $f$ is defined by

$$\sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} c_{jk} e^{i(jx+ky)}, \tag{2.1}$$

where

$$c_{jk} := \frac{1}{4\pi^2} \int_{-\pi}^{\pi} \int_{-\pi}^{\pi} f(u, v)\, e^{-i(ju+kv)}\, du\, dv \qquad (j, k = ..., -1, 0, 1, ...). \tag{2.2}$$

We consider the double sequence of symmetric rectangular partial sums

$$s_{mn}(f, x, y) := \sum_{j=-m}^{m} \sum_{k=-n}^{n} c_{jk} e^{i(jx+ky)} \qquad (m, n = 0, 1, ...). \tag{2.3}$$

In this paper, we assume that $f$ is a function of bounded variation on $\mathbf{T}^2$ in the sense of Hardy and Krause. Then the representation

$$s_{mn}(f, x, y) - s(f, x, y) = \frac{1}{\pi^2} \int_0^{\pi} \int_0^{\pi} \phi_{xy}(u, v)\, D_m(u)\, D_n(v)\, du\, dv \tag{2.4}$$

is an easy consequence of (2.2) and (2.3), where

$$\begin{aligned} s(f, x, y) := \tfrac{1}{4}\{&f(x+0, y+0) + f(x-0, y+0) \\ &+ f(x+0, y-0) + f(x-0, y-0)\}, \end{aligned} \tag{2.5}$$

$$\phi_{xy}(u, v) := \begin{cases} f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) & \\ \quad + f(x-u, y-v) - 4s(f, x, y) & \text{if } u, v > 0; \\ f(x+0, y+v) + f(x-0, y+v) + f(x+0, y-v) & \\ \quad + f(x-0, y-v) - 4s(f, x, y) & \text{if } u = 0 \text{ and } v > 0; \\ f(x+u, y+0) + f(x-u, y+0) + f(x+u, y-0) & \\ \quad + f(x-u, y-0) - 4s(f, x, y) & \text{if } u > 0 \text{ and } v = 0; \\ 0 & \text{if } u = v = 0; \end{cases} \tag{2.6}$$

and

$$D_m(u) := \frac{1}{2} + \sum_{j=1}^{m} \cos ju = \frac{\sin(m+1/2)u}{2\sin u/2} \qquad (m = 0, 1, ...)$$

is the well known Dirichlet kernel. In particular, if $f$ is continuous, then

$$\phi_{xy}(u, v) = f(x+u, y+v) + f(x-u, y+v) + f(x+u, y-v) + f(x-u, y-v) - 4f(x, y).$$

Now, it is not difficult to see that $\phi_{xy}$ is always continuous, especially

$$\lim_{u,v \to +0} \phi_{xy}(u, v) = \lim_{u \to +0} \phi_{xy}(u, 0) = \lim_{v \to +0} \phi_{xy}(0, v) = 0. \tag{2.7}$$

This is the reason that representation (2.4) plays a crucial role in the proofs of Section 5.

Hardy [7] proved the following extension of the Dirichlet–Jordan test (see, e.g., [10, p. 57] from single to double Fourier series.

THEOREM 1. *If $f$ is a function of bounded variation on $\mathbf{T}^2$, $2\pi$-periodic in each variable, then its Fourier series* (2.1) *converges to $s(f, x, y)$ at any point $(x, y)$.*

By convergence we mean the convergence of the symmetric rectangular partial sums $s_{mn}(f, x, y)$ in Pringsheim's sense, i.e., when $m$ and $n$ tend to $\infty$ in (2.3), independently of one another.

Zhizhiashvili [9, p. 223] rediscovered this result with the supplement that if $f$ is continuous on a rectangle $R$, then its Fourier series (2.1) converges to $f(x, y)$ uniformly on any rectangle $R_1$ inside $R$. In addition, he proved that Theorem 1 remains valid if convergence is replaced by $(C, \alpha, \beta)$-summability, where $\alpha, \beta > -1$ are fixed real numbers.

## 3. NEW RESULTS

We begin by recalling the definition of (ordinary) oscillation of a function $h: J \to \mathbf{C}$ over a subinterval $J_1$ of $J$, which reads as

$$\operatorname{osc}_1(h, J_1) := \sup\{|h(t) - h(t')| : t, t' \in J_1\}.$$

Now, we introduce the notion of rectangular oscillation of a function $f: J \times K \to \mathbf{C}$ over a subrectangle $J_1 \times K_1$ of $J \times K$ by setting

$$\operatorname{osc}_2(f, J_1 \times K_1) := \sup\{|f(u, v) - f(u', v) - f(u, v') + f(u', v')| : \\ u, u' \in J_1;\, v, v' \in K_1\}.$$

In the sequel, we will distinguish the subintervals of the nonnegative half of the one-dimensional torus $\mathbf{T} := [-\pi, \pi]$:

$$I_{jm} := \left[\frac{j\pi}{m+1}, \frac{(j+1)\pi}{m+1}\right] \qquad (j = 0, 1, ..., m;\, m = 0, 1, ...).$$

Our first result is a basic estimate of the rate of convergence of the rectangular partial sums of the Fourier series (2.1).

THEOREM 2. *If $f$ is a bounded, measurable function on $\mathbf{T}^2$, $2\pi$-periodic in each variable, such that the four limits $f(x \pm 0, y \pm 0)$ exist at a certain point $(x, y)$, and the four limit functions $f(x \pm 0, \cdot)$ and $f(\cdot, y \pm 0)$ exist, then for any $m, n \geqslant 0$ we have*

$$\begin{aligned} &|s_{mn}(f, x, y) - s(f, x, y)| \\ &\quad \leqslant \left(1 + \frac{2}{\pi} + \frac{1}{\pi^2}\right) \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi_{xy}, I_{jm} \times I_{kn}) \\ &\qquad + \left(1 + \frac{1}{\pi}\right) \sum_{j=0}^{m} \frac{1}{j+1} \operatorname{osc}_1(\phi_{xy}(\cdot, 0), I_{jm}) \\ &\qquad + \left(1 + \frac{1}{\pi}\right) \sum_{k=0}^{n} \frac{1}{k+1} \operatorname{osc}_1(\phi_{xy}(0, \cdot), I_{kn}). \end{aligned} \tag{3.1}$$

We remind the reader that $s(f, x, y)$ and $\phi_{xy}$ are defined in (2.5) and (2.6), respectively.

We note that the counterpart of (3.1) for single Fourier series was proved by Bojanić and Waterman [4].

We will also use the following notations: for functions $f: \mathbf{T}^2 \to \mathbf{C}$, $h: \mathbf{T} \to \mathbf{C}$, and $0 < u, v \leqslant \pi$ we write

$$\begin{aligned} V_2(f, u, v) &:= \operatorname{var}_2(f, [0, u] \times [0, v]), \\ V_1(h, u) &:= \operatorname{var}_1(h, [0, u]) \end{aligned} \tag{3.2}$$

(cf. definitions (1.1) and (1.2)). Now, our second result, which is a particular case of Theorem 2, reads as follows.

THEOREM 3. *Under the conditions of Theorem* 1, *for any* $m, n \geqslant 0$ *we have*

$$\begin{aligned}|s_{mn}(f, x, y) - s(f, x, y)| &\leqslant \frac{4(1+2/\pi+1/\pi^2)}{(m+1)(n+1)} \sum_{j=1}^{m} \sum_{k=1}^{n} V_2\left(\phi_{xy}, \frac{\pi}{j}, \frac{\pi}{k}\right) \\ &+ \frac{2(1+1/\pi)}{m+1} \sum_{j=1}^{m} V_1\left(\phi_{xy}(\cdot, 0), \frac{\pi}{j}\right) \\ &+ \frac{2(1+1/\pi)}{n+1} \sum_{k=1}^{n} V_1\left(\phi_{xy}(0, \cdot), \frac{\pi}{k}\right). \end{aligned} \tag{3.3}$$

Since the continuity of $\phi_{xy}(u, v)$ at $u = v = 0$ implies that

$$\lim_{\delta, \varepsilon \to +0} V_2(\phi_{xy}, \delta, \varepsilon) = \lim_{\delta \to +0} V_1(\phi_{xy}(\cdot, 0), \delta) = \lim_{\varepsilon \to +0} V_1(\phi_{xy}(0, \cdot), \varepsilon) = 0$$

(cf. (2.7)), the right-hand side of inequality (3.3) converges to 0 as $m, n$ tend to $\infty$. In this way, Theorem 1 is an immediate consequence of Theorem 3. Therefore, Theorem 3 can be viewed as a quantitative version of the Dirichlet–Jordan test for double Fourier series.

We note that the counterpart of (3.3) for single Fourier series is due to Bojanić [3].

## 4. AUXILIARY RESULTS

LEMMA 1. *For any* $0 < x \leqslant \pi$ *and* $m \geqslant 0$ *we have*

$$\left| \int_x^\pi D_m(u)\, du \right| \leqslant \frac{\pi}{(m+1)x}. \tag{4.1}$$

We give only a hint of the proof as

$$\int_x^\pi D_m(u)\, du = \frac{\pi - x}{2} - \sum_{j=1}^{m} \frac{\sin jx}{j} = \sum_{j=m+1}^{\infty} \frac{\sin jx}{j},$$

then apply a summation by parts to the right-most series.

For the reader's convenience, we state the relevant results for single Fourier series in the form of Lemmas 2 and 3. To this effect, let $h: \mathbf{T} \to \mathbf{C}$ be a $2\pi$-periodic function such that the two limits

$$\lim_{t \to +0} h(x \pm t) = h(x \pm 0) \tag{4.2}$$

exist. We set

$$\psi_x(t) = \begin{cases} h(x+t)+h(x-t)-h(x+0)-h(x-0) & \text{if } t>0, \\ 0 & \text{if } t=0. \end{cases}$$

LEMMA 2. (Bojanić and Waterman [4]). *If $h$ is a bounded, measurable, and $2\pi$-periodic function such that the limits in* (4.2) *exist at a certain point $x$, then for any $m \geqslant 0$ we have*

$$\left|\frac{1}{\pi}\int_0^\pi \psi_x(t)\, D_m(t)\, dt\right| \leqslant \left(1+\frac{1}{\pi}\right)\sum_{j=0}^{m} \frac{1}{j+1} \operatorname{osc}_1(\psi_x, I_{jm}). \tag{4.3}$$

Actually, Bojanić and Waterman [4] used a weaker version of (4.1) and found (4.3) with the factor $1+2/\pi$ instead of $1+1/\pi$.

LEMMA 3 (Bojanić and Waterman [4]). *If $h$ is a function of bounded variation on $[0, \pi]$, then*

$$\sum_{j=0}^{m} \frac{1}{j+1} \operatorname{osc}_1(h, I_{jm}) \leqslant \frac{2}{m+1} \sum_{j=1}^{m} V_1\left(h, \frac{\pi}{j}\right). \tag{4.4}$$

Actually, inequality (4.4) was proved in [4] in a more general setting, namely for functions of generalized bounded variation.

## 5. PROOFS

*Proof of Theorem* 2. We start with representation (2.4) by writing $\phi$ instead of $\phi_{xy}$. It is plain that

$$\begin{aligned} &s_{mn}(f, x, y) - s(f, x, y) \\ &\quad = \frac{1}{\pi^2}\int_0^\pi \int_0^\pi \{\phi(u, v) - \phi(u, 0) - \phi(0, v)\}\, D_m(u)\, D_n(v)\, du\, dv \\ &\qquad + \frac{1}{2\pi}\int_0^\pi \phi(u, 0)\, D_m(u)\, du \\ &\qquad + \frac{1}{2\pi}\int_0^\pi \phi(0, v)\, D_n(v)\, dv \\ &\quad =: A_{mn} + B_m + C_n, \qquad \text{say.} \end{aligned} \tag{5.1}$$

First, we treat $A_{mn}$. To this effect, we introduce the auxiliary function

$$g(u, v) := \phi(u, v) - \phi(u, 0) - \phi(0, v). \tag{5.2}$$

Clearly, $g(u, v)$ is also continuous and, in addition, for any $u, v > 0$

$$g(u, 0) = g(0, v) = g(0, 0) = 0.$$

Besides, we will use the notation

$$\theta_{jm} := \frac{j\pi}{m+1} \qquad (j = 0, 1, ..., m;\ m = 0, 1, ...).$$

Then $I_{jm} = [\theta_{jm}, \theta_{j+1,m}]$.

We decompose the double integral defining $A_{mn}$ as

$$\begin{aligned}
\pi^2 A_{mn} &= \int_{I_{0m}} \int_{I_{0n}} g(u, v)\, D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{j=1}^{m} \int_{I_{jm}} \int_{I_{0n}} \{g(u, v) - g(\theta_{jm}, v)\}\, D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{j=1}^{m} \int_{I_{jm}} \int_{I_{0n}} g(\theta_{jm}, v)\, D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{k=1}^{n} \int_{I_{0m}} \int_{I_{kn}} \{g(u, v) - g(u, \theta_{kn})\}\, D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{k=1}^{n} \int_{I_{0m}} \int_{I_{kn}} g(u, \theta_{kn})\, D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{I_{jm}} \int_{I_{kn}} \{g(u, v) - g(\theta_{jm}, v) - g(u, \theta_{kn}) + g(\theta_{jm}, \theta_{kn})\} \\
&\quad \times D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{I_{jm}} \int_{I_{kn}} \{g(\theta_{jm}, v) - g(\theta_{jm}, \theta_{kn})\}\, D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{I_{jm}} \int_{I_{kn}} \{g(u, \theta_{kn}) - g(\theta_{jm}, \theta_{kn})\}\, D_m(u)\, D_n(v)\, du\, dv \\
&\quad + \sum_{j=1}^{m} \sum_{k=1}^{n} \int_{I_{jm}} \int_{I_{kn}} g(\theta_{jm}, \theta_{kn})\, D_m(u)\, D_n(v)\, du\, dv \\
&=: A_1 + A_2 + \cdots + A_9, \qquad \text{say.}
\end{aligned} \tag{5.3}$$

In the sequel, we frequently use the inequalities

$$|D_m(u)| \leqslant \min\left\{m + \frac{1}{2}, \frac{\pi}{2u}\right\} \qquad \text{for} \quad 0 < u \leqslant \pi. \tag{5.4}$$

By this and (5.2),

$$|A_1| \leqslant \operatorname{osc}_2(\phi, I_{0m} \times I_{0n}) \int_{I_{0m}} \int_{I_{0n}} \left(m+\frac{1}{2}\right)\left(n+\frac{1}{2}\right) du\, dv$$
$$\leqslant \pi^2 \operatorname{osc}_2(\phi, I_{0m} \times I_{0n}). \tag{5.5}$$

By definition,

$$g(u, v) - g(\theta_{jm}, v) = \phi(u, v) - \phi(\theta_{jm}, v) - \phi(u, 0) + \phi(\theta_{jm}, 0).$$

Thus, by (5.4),

$$|A_2| \leqslant \sum_{j=1}^{m} \operatorname{osc}_2(\phi, I_{0m} \times I_{0n}) \int_{I_{jm}} \int_{I_{0n}} \frac{\pi}{2\theta_{jm}} \left(n+\frac{1}{2}\right) du\, dv$$
$$\leqslant \pi \sum_{j=1}^{m} \frac{1}{2j} \operatorname{osc}_2(\phi, I_{jm} \times I_{0n})$$
$$\leqslant \pi \sum_{j=1}^{m} \frac{1}{j+1} \operatorname{osc}_2(\phi, I_{jm} \times I_{0n}). \tag{5.6}$$

Setting

$$R_{jm} := \int_{\theta_{jm}}^{\pi} D_m(u)\, du, \tag{5.7}$$

by virtue of Lemma 1, we have

$$|R_{jm}| \leqslant \frac{1}{j} \qquad \left(j = 1, 2, ..., m;\ R_{0m} = \frac{\pi}{2},\ R_{m+1,m} = 0\right). \tag{5.8}$$

Performing a summation by parts gives

$$A_3 = \int_{I_{0n}} \left\{ \sum_{j=1}^{m} g(\theta_{jm}, v)(R_{jm} - R_{j+1,m}) \right\} D_n(v)\, dv$$
$$= \int_{I_{0n}} \left\{ \sum_{j=1}^{m} (g(\theta_{jm}, v) - g(\theta_{j-1,m}, v))\, R_{jm} \right\} D_n(v)\, dv. \tag{5.9}$$

Since, by (5.2),

$$g(\theta_{jm}, v) - g(\theta_{j-1,m}, v) = \phi(\theta_{jm}, v) - \phi(\theta_{j-1,m}, v) - \phi(\theta_{jm}, 0) + \phi(\theta_{j-1,m}, 0),$$

by (5.4), (5.8), and (5.9), we conclude that

$$
\begin{aligned}
|A_3| &\leqslant \sum_{j=1}^{m} \frac{1}{j} \operatorname{osc}_2(\phi, I_{j-1,m} \times I_{0n}) \int_{I_{0n}} \left(n+\frac{1}{2}\right) dv \\
&\leqslant \pi \sum_{j=0}^{m-1} \frac{1}{j+1} \operatorname{osc}_2(\phi, I_{jm} \times I_{0n}).
\end{aligned} \tag{5.10}
$$

Analogously, we can see that

$$
|A_4| \leqslant \pi \sum_{k=1}^{n} \frac{1}{k+1} \operatorname{osc}_2(\phi, I_{0m} \times I_{kn}) \tag{5.11}
$$

and

$$
|A_5| \leqslant \pi \sum_{k=0}^{n-1} \frac{1}{k+1} \operatorname{osc}_2(\phi, I_{0m} \times I_{kn}). \tag{5.12}
$$

By (5.2), we have

$$
\begin{aligned}
&g(u, v) - g(\theta_{jm}, v) - g(u, \theta_{kn}) + g(\theta_{jm}, \theta_{kn}) \\
&\quad = \phi(u, v) - \phi(\theta_{jm}, v) - \phi(u, \theta_{kn}) + \phi(\theta_{jm}, \theta_{kn}),
\end{aligned}
$$

and, by (5.4), estimate as

$$
\begin{aligned}
|A_6| &\leqslant \sum_{j=1}^{m} \sum_{k=1}^{n} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}) \int_{I_{jm}} \int_{I_{kn}} \frac{\pi^2}{4\theta_{jm}\theta_{kn}} \, du \, dv \\
&\leqslant \frac{\pi^2}{4} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{jk} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}) \\
&\leqslant \pi^2 \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}).
\end{aligned} \tag{5.13}
$$

Performing a single summation by parts, while using notation (5.7), we obtain that

$$
\begin{aligned}
A_7 &= \sum_{k=1}^{n} \int_{I_{kn}} \left\{ \sum_{j=1}^{m} (g(\theta_{jm}, v) - g(\theta_{jm}, \theta_{kn}))(R_{jm} - R_{j+1,m}) \right\} D_n(v) \, dv \\
&= \sum_{k=1}^{n} \int_{I_{kn}} \left\{ \sum_{j=1}^{m} (g(\theta_{jm}, v) - g(\theta_{jm}, \theta_{kn}) - g(\theta_{j-1,m}, v) \right. \\
&\qquad \left. + g(\theta_{j-1,m}, \theta_{kn}))\, R_{jm} \right\} D_n(v) \, dv.
\end{aligned} \tag{5.14}
$$

Since, by (5.2),

$$g(\theta_{jm}, v) - g(\theta_{jm}, \theta_{kn}) - g(\theta_{j-1,m}, v) + g(\theta_{j-1,m}, \theta_{kn})$$
$$= \phi(\theta_{jm}, v) - \phi(\theta_{jm}, \theta_{kn}) - \phi(\theta_{j-1,m}, v) + \phi(\theta_{j-1,m}, \theta_{kn}),$$

from (5.8) and (5.14) it follows that

$$\begin{aligned} |A_7| &\leqslant \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{j} \operatorname{osc}_2(\phi, I_{j-1,m} \times I_{kn}) \int_{I_{kn}} \frac{\pi}{2\theta_{kn}} \, dv \\ &\leqslant \frac{\pi}{2} \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{jk} \operatorname{osc}_2(\phi, I_{j-1,m} \times I_{kn}) \\ &\leqslant \pi \sum_{j=0}^{m-1} \sum_{k=1}^{n} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}). \end{aligned} \tag{5.15}$$

Similarly, we can find that

$$|A_8| \leqslant \pi \sum_{j=1}^{m} \sum_{k=0}^{n-1} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}). \tag{5.16}$$

Keeping notation (5.7) in mind, we may write

$$A_9 = \sum_{j=1}^{m} \sum_{k=1}^{n} g(\theta_{jm}, \theta_{kn})(R_{jm} - R_{j+1,m})(R_{kn} - R_{k+1,n}),$$

whence a double summation by parts gives

$$\begin{aligned} A_9 &= \sum_{j=1}^{m} \sum_{k=1}^{n} \{ g(\theta_{jm}, \theta_{kn}) - g(\theta_{j-1,m}, \theta_{kn}) - g(\theta_{jm}, \theta_{k-1,n}) \\ &\quad + g(\theta_{j-1,m}, \theta_{k-1,n}) \} R_{jm} R_{kn} \\ &= \sum_{j=1}^{m} \sum_{k=1}^{n} \{ \phi(\theta_{jm}, \theta_{kn}) - \phi(\theta_{j-1,m}, \theta_{kn}) - \phi(\theta_{jm}, \theta_{k-1,n}) \\ &\quad + \phi(\theta_{j-1,m}, \theta_{k-1,n}) \} R_{jm} R_{kn} \end{aligned}$$

(cf. (5.2)). Thus, from (5.8) it follows that

$$\begin{aligned} |A_9| &\leqslant \sum_{j=1}^{m} \sum_{k=1}^{n} \frac{1}{jk} \operatorname{osc}_2(\phi, I_{j-1,m} \times I_{k-1,n}) \\ &= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}). \end{aligned} \tag{5.17}$$

Combining (5.3), (5.5), (5.6), (5.10)–(5.13), (5.15–(5.17) yields

$$|A_{mn}| \leqslant \left(1+\frac{2}{\pi}+\frac{1}{\pi^2}\right) \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}). \quad (5.18)$$

In order to estimate $B_m$ and $C_n$ in (5.1), it is enough to apply Lemma 2, which gives

$$|B_m| \leqslant \left(1+\frac{1}{\pi}\right) \sum_{j=0}^{m} \frac{1}{j+1} \operatorname{osc}_1(\phi(\cdot, 0), I_{jm}) \quad (5.19)$$

and

$$|C_n| \leqslant \left(1+\frac{1}{\pi}\right) \sum_{k=0}^{n} \frac{1}{k+1} \operatorname{osc}_1(\phi(0, \cdot), I_{kn}). \quad (5.20)$$

Now, combining (5.1), (5.18)–(5.20) results in (3.1) to be proved.

*Proof of Theorem* 3. We fix $m$ and $n$, and set

$$M_{jk} := \sum_{i=0}^{j} \sum_{l=0}^{k} \operatorname{osc}_2(\phi, I_{im} \times I_{ln}) \qquad (j=0, 1, ..., m; k=0, 1, ..., n). \quad (5.21)$$

We also define a function $M(u, v)$ on the rectangle $[\pi/(m+1), \pi) \times [\pi/(n+1), \pi)$ by

$$M(u, v) := M_{jk} \qquad \text{for} \quad (u, v) \in [\theta_{j+1,m}, \theta_{j+2,m}) \times [\theta_{k+1,n}, \theta_{k+2,n}),$$

which is $I_{j+1,m} \times I_{k+1,n}$ apart from the top segment and the right segment bordering the rectangle, and for $j=0, 1, ..., m-1$; $k=0, 1, ..., n-1$. Clearly,

$$\operatorname{osc}_2(\phi, I_{jm} \times I_{kn}) = \begin{cases} M_{00} \\ \qquad \text{if} \quad j=k=0, \\ M_{j0} - M_{j-1,0} \\ \qquad \text{if} \quad j \geqslant 1 \quad \text{and} \quad k=0, \\ M_{0k} - M_{0,k-1} \\ \qquad \text{if} \quad j=0 \quad \text{and} \quad k \geqslant 1, \\ M_{jk} - M_{j-1,k} - M_{j,k-1} + M_{j-1,k-1} \\ \qquad \text{if} \quad j, k \geqslant 1. \end{cases}$$

A double summation by parts shows that

$$\begin{aligned}
\Sigma_{mn} &:= \sum_{j=0}^{m} \sum_{k=0}^{n} \frac{1}{(j+1)(k+1)} \operatorname{osc}_2(\phi, I_{jm} \times I_{kn}) \\
&= \sum_{j=0}^{m-1} \sum_{k=0}^{n-1} M_{jk} \left(\frac{1}{j+1} - \frac{1}{j+2}\right)\left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\
&\quad + \frac{1}{n+1} \sum_{j=0}^{m-1} M_{jn} \left(\frac{1}{j+1} - \frac{1}{j+2}\right) \\
&\quad + \frac{1}{m+1} \sum_{k=0}^{n-1} M_{mk} \left(\frac{1}{k+1} - \frac{1}{k+2}\right) \\
&\quad + \frac{M_{mn}}{(m+1)(n+1)}. \qquad (5.22)
\end{aligned}$$

We will use very simple properties of the two-dimensional Riemann–Stieltjes integral. We rely on the facts that $M(u, v)$ takes on constant values over a finite number of nonoverlapping rectangles with sides parallel to the coordinate axes and that the functions $(-1/u)$ and $(-1/v)$ are continuous and nondecreasing for $u, v > 0$. Therefore, the right-hand side of (5.22) can be rewritten as

$$\begin{aligned}
\Sigma_{mn} &= \frac{\pi}{(m+1)(n+1)} \int_{\theta_{1m}}^{\pi} \int_{\theta_{1n}}^{\pi} M(u, v)\, d\left(-\frac{1}{u}\right)\left(-\frac{1}{v}\right) \\
&\quad + \frac{\pi}{(m+1)(n+1)} \int_{\theta_{1m}}^{\pi} M(u, \theta_{n,n+1})\, d\left(-\frac{1}{u}\right) \\
&\quad + \frac{\pi}{(m+1)(n+1)} \int_{\theta_{1n}}^{\pi} M(\theta_{m,m+1}, v)\, d\left(-\frac{1}{v}\right) \\
&\quad + \frac{M_{mn}}{(m+1)(n+1)}. \qquad (5.23)
\end{aligned}$$

First, we consider two arbitrary partitions

$$\theta_{1m} = a_p < a_{p-1} < \cdots < a_0 = \pi \qquad \text{and} \qquad \theta_{1n} = b_q < b_{q-1} < \cdots < b_0 = \pi,$$

where $p$ and $q$ are positive integers. By (5.23), we obviously have that

$$\begin{aligned}
\Sigma_{mn} &\leqslant \frac{1}{(m+1)(n+1)} \\
&\quad \times \left\{ \pi^2 \sum_{j=0}^{p-1} \sum_{k=0}^{q-1} M(a_j, b_k) \left(\frac{1}{a_{j+1}} - \frac{1}{a_j}\right)\left(\frac{1}{b_{k+1}} - \frac{1}{b_k}\right) \right. \\
&\quad + \pi \sum_{j=0}^{p-1} M(a_j, \theta_{n,n+1}) \left(\frac{1}{a_{j+1}} - \frac{1}{a_j}\right) \\
&\quad \left. + \pi \sum_{k=0}^{q-1} M(\theta_{m,m+1}, b_k) \left(\frac{1}{b_{k+1}} - \frac{1}{b_k}\right) + M_{mn} \right\}. \qquad (5.24)
\end{aligned}$$

Taking into account definitions (1.1), (1.2), and notations (3.2), (5.21), we infer in turn that

$$
\begin{aligned}
M(a_j, b_k) &\leqslant \operatorname{var}_2(\phi[0, a_j] \times [0, b_k]) = V_2(\phi, a_j, b_k),\\
M(a_j, \theta_{n,n+1}) &\leqslant \operatorname{var}_2(\phi, [0, a_j] \times [0, \pi]) = V_2(\phi, a_j, \pi),\\
M(\theta_{m,m+1}, b_k) &\leqslant V_2(\phi, \pi, b_k),\\
M_{mn} &\leqslant V_2(\phi, \pi, \pi).
\end{aligned}
$$

Second, we choose $p := m$, $q := n$,

$$a_j := \frac{\pi}{j+1} \quad \text{and} \quad b_k := \frac{\pi}{k+1} \qquad (j = 0, 1, ..., m;\, k = 0, 1, ..., n).$$

From (5.24) it follows that

$$
\begin{aligned}
\sum\nolimits_{mn} &\leqslant \frac{1}{(m+1)(n+1)}\\
&\quad \times \left\{ \sum_{j=1}^{m} \sum_{k=1}^{n} V_2\left(\phi, \frac{\pi}{j}, \frac{\pi}{k}\right) + \sum_{j=1}^{m} V_2\left(\phi, \frac{\pi}{j}, \pi\right)\right.\\
&\quad \left. + \sum_{k=1}^{n} V_2\left(\phi, \pi, \frac{\pi}{k}\right) + V_2(\phi, \pi, \pi)\right\}\\
&\leqslant \frac{4}{(m+1)(n+1)} \sum_{j=1}^{m} \sum_{k=1}^{n} V_2\left(\phi, \frac{\pi}{j}, \frac{\pi}{k}\right). \qquad (5.25)
\end{aligned}
$$

Third, it remains to apply Lemma 3 in order to obtain the two estimates

$$\sum_{j=0}^{m} \frac{1}{j+1} \operatorname{osc}_1(\phi(\cdot, 0), I_{jm}) \leqslant \frac{2}{m+1} \sum_{j=1}^{m} V_1\left(\phi(\cdot, 0), \frac{\pi}{j}\right) \qquad (5.26)$$

and

$$\sum_{k=0}^{n} \frac{1}{k+1} \operatorname{osc}_1(\phi(0, \cdot), I_{kn}) \leqslant \frac{2}{n+1} \sum_{k=1}^{n} V_1\left(\phi(0, \cdot), \frac{\pi}{k}\right). \qquad (5.27)$$

Substituting (5.25)–(5.27) (see the definition of $\sum_{mn}$ in (5.22)) into the right-hand side of (3.1), we conclude (3.3) to be proved.

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